p.0

① Finding a starting BFS

② Entering variable — optimality condition

non basic variable $\longrightarrow$ basic variable

current 0

③ Leaving variable $\boxed{\text{feasible condition}}$

basic variable $\longrightarrow$ non basic

≥ 0 $= 0$

④ move from ~~one~~ current BFS $\to$ next BFS

(Gaussian Elimination)

Assume that x_j (non basic) is entering.

X_{B_r} to be leaving which r.

$Y = B^{-1}A \Leftrightarrow \underset{m\times n}{A} = \underset{m\times m}{B}\ \underset{m\times n}{Y}$ B = basic matrix

$= [b_1, \ldots, b_m]$

$= [\vec{a}_{i_1} \ldots \vec{a}_{i_m}]$

$[\vec{y}_1 \ldots \vec{y}_j \quad \vec{y}_n]$

$\vec{y}_j$ is the entering

$\vec{y}_j = \begin{pmatrix} y_{1j} \\ \vdots \\ y_{mj} \end{pmatrix}$,

feasibility test

$\min_{1\le i\le m} \left\{ \frac{x_{B_i}}{y_{ij}} : y_{ij} > 0 \right\}$

current basic solution

$\downarrow$ r the leaving variable

$y_{rj} > 0$

ensure that the next solution is feasible

$$\max f(\vec{x}) = \vec{c}^{\,T}\vec{x} \leftarrow \text{linear}$$

$$\text{s.t.} \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq 0 \end{cases}$$

$$f(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x}-\vec{x}_0) + \text{h.o.t.}$$

$$\Longleftrightarrow \quad f(\vec{x}) = f(\vec{x}_0) + \boxed{\nabla f(\vec{x}_0)(\vec{x}-\vec{x}_0)} \; * \quad (\because \text{linear})$$

$$f(\vec{x}) = f(\vec{x}_0) + \vec{c}^{\,T}(\vec{x}-\vec{x}_0)$$

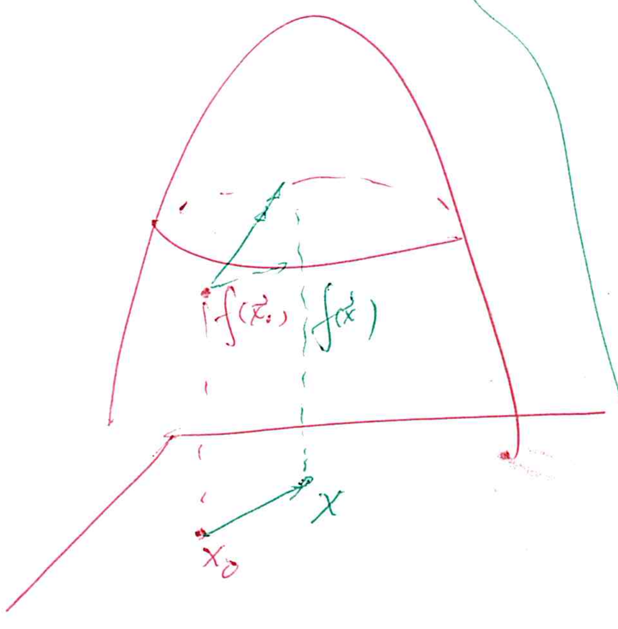
$$\vec{c}^{\,T}\vec{x} = \vec{c}^{\,T}\vec{x}_0 + \underbrace{\vec{c}^{\,T}(\vec{x}-\vec{x}_0)}_{\text{is } > 0}$$

$$\Downarrow$$

$$f(\vec{x}) > f(\vec{x}_0)$$

$\nabla f(\vec{x}_0)(\vec{x}-\vec{x}_0)$

is directional derivative of f at $\vec{x}_0$ in direction of $\vec{x}-\vec{x}_0$,



Choose $\vec{x}$ s.t.

$$\nabla f(\vec{x}_0)(\vec{x}-\vec{x}_0) > 0$$

$$\Longleftrightarrow \vec{c}^{\,T}(\underset{\text{next BFS}}{\vec{x}} - \underset{\text{Current BFS}}{\vec{x}_0}) > 0.$$

P2

current

$$A = [\, B \mid R \,] \qquad \vec{x} = \begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix}$$

x_j is entering & x_{Br} is leaving

$$\underset{\vec{x}_0}{\overset{\text{current}}{\begin{pmatrix} x_{B1} \\ x_{Br} \\ x_{Bm} \\ \hline 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}}}
\begin{array}{l} \\ \\ m \\ \\ \\ \leftarrow j \\ \\ \\ \end{array}
\qquad
\underset{\vec{x}}{\overset{\text{next}}{\begin{pmatrix} x_{B1} - x_{Br}\frac{y_{1j}}{y_{rj}} \\ \vdots \\ 0 \\ x_{Bm} - x_{Br}\frac{y_{mj}}{y_{rj}} \\ \hline 0 \\ \vdots \\ x_{Br}/y_{rj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}}
\begin{array}{l} \\ \\ \leftarrow r \\ m \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}$$

directional = derivative

$$\vec{c}^{\,T}(\vec{x} - \vec{x}_0) = \vec{c}^{\,T}
\begin{pmatrix} -x_{Br}\frac{y_{1j}}{y_{rj}} \\ \vdots \\ -x_{Br} \\ -x_{Br}\frac{y_{mj}}{y_{rj}} \\ \hline 0 \\ \vdots \\ x_{Br}/y_{rj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\begin{array}{l} \leftarrow 1 \\ \\ \leftarrow r \\ \leftarrow m \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}$$

$$\vec{c}^{\,T} = (\vec{c}_1 \cdots \vec{c}_n)$$

$$= (c_{B_1}, \ldots, c_{B_m}, * * \cdots c_j, * * \cdots *)$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{B_i}\left(-x_{Br}\frac{y_{ij}}{y_{rj}}\right) - c_{B_r} \cdot x_{B_r}\left(\frac{y_{rj}}{y_{rj}}\right) + c_j \cdot \frac{x_{Br}}{y_{rj}}$$

$$= -\sum_{i=1}^{m} c_{B_i} x_{Br}\left(\frac{y_{ij}}{y_{rj}}\right) + c_j\left(\frac{x_{Br}}{y_{rj}}\right)$$

$$= \frac{x_{Br}}{y_{rj}}\left\{ c_j - \underbrace{\sum_{i=1}^{m} c_{B_i} y_{ij}}_{z_j} \right\} = \frac{x_{Br}}{y_{rj}}\{c_j - z_j\}$$

$$f(\vec{x}) = f(\vec{x}_0) + \text{d.d in } x_j \text{ direction}$$

$$= f(\vec{x}_0) + \frac{x_{Br}}{y_{rj}}\{c_j - z_j\}$$

$x_{Br} > 0$

$y_{rj} > 0$ (feasibility condition) in the next step.

Current objective value

Optimality Condition

$$\boxed{\text{Choose } j \ (x_j) \text{ s.t. } c_j - z_j > 0}$$

$$z_j = \sum_{i=1}^{m} c_{B_i} y_{ij}.$$

$$\Leftrightarrow (z_1, \ldots, z_n) = \underbrace{(c_{B_1}, \ldots, c_{B_m})}\left[\ Y\ \right]$$

$(\vec{c}^T - \vec{z}^T)$ choose j s.t. the entry is > 0.

$$\boxed{\vec{z}^T = \vec{c}_B^T Y}$$

P4

Since $\mathbf{x}$ is a feasible solution, $A\mathbf{x} = \mathbf{b}$. Thus by (2.24),

$$\mathbf{b} = \sum_{j=1}^{n} x_j \mathbf{a}_j = \sum_{j=1}^{n} x_j (B\mathbf{y}_j) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_{ij} \mathbf{b}_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} y_{ij} x_j \right) \mathbf{b}_i = \sum_{i=1}^{m} \tilde{x}_i \mathbf{b}_i = B\tilde{\mathbf{x}}.$$

Since B is non-singular and we already have $B\mathbf{x}_B = \mathbf{b}$, it follows that $\tilde{\mathbf{x}} = \mathbf{x}_B$. Thus by (2.27),

$$z \leq \sum_{i=1}^{m} x_{B_i} c_{B_i} = z_0$$

for all $\mathbf{x}$ in the feasible region. □

Example 2.4. Let us consider the LPP with

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{c}^T = [2, 5, 6, 8] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Let us choose our starting B as

$$B = [\mathbf{a}_1, \mathbf{a}_4] = [\mathbf{b}_1, \mathbf{b}_2] = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}.$$

Then it is easily checked that the corresponding basic solution is $\mathbf{x}_B = [1, 1]^T$, which is clearly feasible with objective value

$$z = \mathbf{c}_B^T \mathbf{x}_B = [2, 8] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 10.$$

Since

$$B^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix},$$

$A = BY$

$Y = B^{-1}A$

$Z^T = C_B^T Y$

by (2.24), the y_{ij} are given by

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{y}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \mathbf{y}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence

$$z_2 = [2, 8] \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = 4,$$

and

$$z_3 = [2, 8] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 6.$$

Since $z_2 - c_2 = -1$ and $z_3 - c_3 = 0$, we see that $\mathbf{a}_2$ is the *entering* column. As remarked above,

$$z_1 - c_1 = z_4 - c_4 = 0,$$

because x_1 and x_4 are basic variables. Looking at the column entries of $\mathbf{y}_2$, we find that y_{22} is the only positive entry. Hence $\mathbf{b}_2 = \mathbf{a}_4$ is the *leaving* column. Thus

$$\hat{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

and the corresponding basic solution is found to be $\hat{\mathbf{x}}_B = [2, \frac{3}{2}]^T$, which is clearly feasible as expected. The new objective value is given by

$$\hat{z} = [2, 5] \begin{bmatrix} 2 \\ \frac{3}{2} \end{bmatrix} = 11.5 > z.$$

$$\max f = 2x_1 + 5x_2 + 6x_3 + 8x_4$$

s.t.
$$\begin{cases} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \\ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ x_1, \dots, x_4 \geq 0 \end{cases}$$

$$B = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

① BFS $\quad f = 2\cdot 1 + 8 \cdot 1 = 10$

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$x_1, x_4 \geq 0$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{array}{c} x_1 \\ x_4 \end{array}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Starting BFS.

②

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4) = A = BY$$

$$= [\vec{a}_1, \vec{a}_4]\left[\begin{smallmatrix}1\\0\end{smallmatrix}, *, *, \begin{smallmatrix}0\\1\end{smallmatrix}\right]$$

$$Y = B^{-1}A$$

$$= \begin{bmatrix} 1 & -2/3 & -1 & 0 \\ 0 & 2/3 & 1 & 1 \end{bmatrix}$$

Corresponding to the basic variables Y's columns $= I$.

$$\vec{C}_B = (C_1, C_4) = (2 \quad 8)$$

$$\vec{z} = (2, 8)\cdot \underset{\uparrow}{Y} = (2, 4, 6, 8)$$

$$\vec{C}^T - \vec{z}^T = (2\ 5\ 6\ 8) - (2\ 4\ 6\ 8)$$

$$= (0, 1, 0, 0) \quad \boxed{\text{if } \quad c_2 - z_2 > 0}$$

optimality condition

$\boxed{x_2 \text{ is the entering variable}}$

③

$$Y = \begin{pmatrix} 1 & -2/3 & -1 & 0 \\ 0 & 2/3 & 1 & 1 \end{pmatrix}$$

leaving variable

entering x_2

$$\min_{1 \leq i \leq 2}\left\{ \frac{x_{Br}}{y_{r2}} : y_{r2} > 0 \right\} \quad \left\{ \frac{x_1}{y_{12}}, \frac{x_4}{y_{22}}, \ y_{\ast\ast} > 0 \right\}$$

x_2 is entering

$$\min\left\{ \cancel{\frac{1}{-2/3}}\ {}_{<0}, \ \frac{1}{2/3} \right\}$$

x_1 $\quad x_4 \to$ leaving variable

④

$$\text{BFS} \begin{pmatrix} x_1 = 1 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \geq 0 \\ x_2 > 0 \\ x_3 = 0 \\ x_4 = 0 \end{pmatrix}$$

P6

Since

$$\hat{B}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix},$$

by (2.24), the $\hat{y}_{ij}$ are given by

$$\hat{\mathbf{y}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{\mathbf{y}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \hat{\mathbf{y}}_3 = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} \quad \hat{\mathbf{y}}_4 = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}.$$

$A = B_1 Y_1$

$\vec{z}_1^{\,T} = \vec{c}_B Y_1$

Hence

$$z_3 - c_3 = [2, 5]\begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} - 6 = 1.5,$$

and

$$z_4 - c_4 = [2, 5]\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} - 8 = 1.5.$$

Since all $z_j - c_j \geq 0$, $1 \leq j \leq 4$, we see that the point $[2, \frac{3}{2}]$ is an optimal solution.

The last example illustrates how one can find the optimal solution by searching through the basic feasible solutions. That is exactly what *simplex method* does. However, the simplex method uses tableaus to minimize the book-keeping work that we encountered in the last example.

next time x_1, x_2 are basic
$\vec{a}_1$ $\vec{a}_2$

$$\hat{B} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \hat{B}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \hat{B}^{-1}\vec{b} = \begin{pmatrix} 2 \\ 3/2 \end{pmatrix} \Rightarrow$$

Solution $\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = \begin{pmatrix} 2 \\ 3/2 \\ 0 \\ 0 \end{pmatrix}$

$$f = 2\cdot 2 + 5\cdot 3/2 = 4 + \frac{15}{2} = 11.5$$

$$A = \hat{B}\hat{Y}$$

$$\hat{Y} = \hat{B}^{-1}A = \left[\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}\ \boxed{\begin{matrix} 0 & 1 \\ 3/2 & 3/2 \end{matrix}}\right]$$

$x_4 = 1$ $= 10 + 1.5$

$1.5 \stackrel{?}{=} \frac{x_{B r}}{y_{rj}}(c_j - z_j)$, $y_{rj} = 2/3$, $c_j - z_j = 1$, $\frac{1}{\frac{2}{3}}\cdot 1 = 3$

$$\hat{Z} = \hat{C}_B^T\hat{Y} = (\underbrace{2 \quad 5}_{c_1 \quad c_2})\hat{Y} = (2 \quad 5 \quad \tfrac{15}{2} \quad \tfrac{19}{2})$$

$\hat{C}_B$: new "current"

c_1, c_2: current basis

$$\hat{C}^T - \hat{Z}^T = (2\ 5\ 6\ 8) - (2\ 5\ \tfrac{15}{2}\ \tfrac{19}{2})$$

$$= (0,\ 0,\ -1.5,\ -1.5) \Rightarrow \text{"optimal" of LPP}$$

None are positive $\Rightarrow$ x_3 direct$\downarrow$ x_4 direct$\downarrow$

"objective is linear"

Thm 2.7

$c_j - z_j \leq 0 \quad \forall j \Rightarrow$ then we are at the optimum

$[B \mid R] \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix} = \vec{b} \Leftrightarrow B\vec{x}_B = \vec{b} \Leftrightarrow \vec{x}_B = B^{-1}\vec{b}$

pf: ① $A \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix} = \vec{b}$, $\vec{x}_B \geq 0$ ③

$\vec{c}_B^{\,T} \vec{x}_B = \vec{c}^{\,T} \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix}$ = current objective value

② $\forall \vec{u} \in FR \begin{pmatrix} A\vec{u} = \vec{b} \\ \vec{u} \geq \vec{0} \end{pmatrix}$ $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

$\overset{??}{\Rightarrow} \boxed{\vec{c}^{\,T} \begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix} \geq \vec{c}^{\,T} \vec{u}}$

$\forall \vec{u} \in FR$: $\vec{c}^{\,T} \vec{u} = \sum_{i=1}^{n} c_i u_i \leq \sum_{j=1}^{n} z_j u_j$ (by assumption)

$= \vec{z}^{\,T} \vec{u}$

$= \cancel{\vec{z}^{\,T} A}$ $\quad \boxed{\vec{z}^{\,T} = \vec{c}_B^{\,T} Y}$

$= \vec{c}_B^{\,T} Y \vec{u}$ $\quad \boxed{A = BY}$

$= \vec{c}_B^{\,T} B^{-1} A \vec{u}$ $\quad (A u = \vec{b})$

$= \vec{c}_B^{\,T} B^{-1} \vec{b}$ $\quad (B^{-1}\vec{b} = \vec{x}_B)$

$= \vec{c}_B^{\,T} \vec{x}_B$ ///

Ex. 3.2

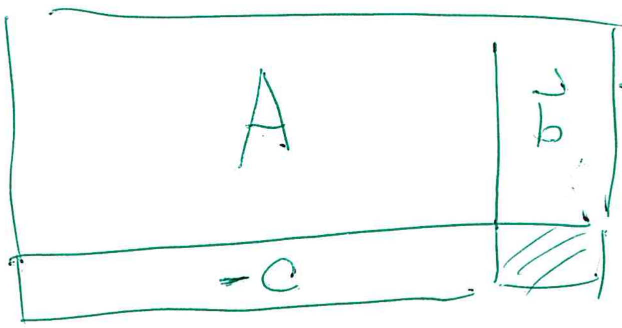
$$\text{Max} \quad x_0 = 3x_1 + x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\begin{cases} \begin{pmatrix} 2 & 1 & 1 & \overbrace{\begin{matrix} 1 & 0 & 0 \end{matrix}}^{= I \text{ invertible}} \\ 1 & 2 & 3 & \begin{matrix} 0 & 1 & 0 \end{matrix} \\ 2 & 2 & 1 & \begin{matrix} 0 & 0 & 1 \end{matrix} \end{pmatrix}_{3\times 6} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \\ \\ x_1, \dots x_6 \geq 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \\ \boxed{\begin{matrix} 2 \\ 5 \\ 6 \end{matrix}} \end{pmatrix} \geq 0$$

non-base variable

Base variable



Simplex tableau